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DEGENERATE BIFURCATION GIVING RISE TO A CYCLE IN MULTIPARAMETER PROBLEMS IN HYDRODYNAMICS†

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An algorithm for constructing the Lyapunov–Schmidt series in multiparameter problems with quadratic non-linearity is proposed, which enables a degenerate bifurcation of cycle generation to be investigated using the Weierstrass preparation theorem. The use of this algorithm to study multiparameter problems of hydrodynamics, namely, Kolmogorov flow and the Couette–Poiseuille flow in a plane channel is considered.

One of the methods of studying bifurcations which generate to a cycle (the Lyapunov–Schmidt method) was applied independently by a small number of researchers [1–3] to the problem of the onset of self-induced oscillations in a fluid. Later it was shown [4, 5] how to obtain an inductive construction of expansions in the subcritical parameter $\varepsilon = \theta(R - R_0)^{1/2}$, $\theta = \pm 1$, where R_0 is the critical Reynolds number corresponding to the stability loss in the main flow. A constructive procedure for constructing such series was presented in [6, 7] and these expansions were used to study the stability loss in Poiseuille flow.

However, expansions in ε can be constructed only under non-degeneracy conditions, which in multiparameter problems may be violated on a certain manifold of unity codimension in parameter space.

It is well known [1] that expansions in powers of the amplitude are more general. They can be used to study branching solutions in the neighbourhood of a degenerate point. However, in this and subsequent papers insufficient attention was devoted to the algorithmic aspect of constructing such expansions because for non-linearities of general form the problem is intractable.

The purpose of the paper is to present explicit formulae which enable the expansions to be constructed in problems of hydrodynamics, and to show how these results can be applied to the Kolmogorov and Couette–Poiseuille problems.

1. APPLICATIONS OF THE LYAPUNOV–SCHMIDT REDUCTION METHOD TO THE STUDY OF THE ONSET OF SELF-INDUCED OSCILLATIONS IN A FLUID

We recall briefly, making no pretensions to originality, the well-known construction of the Lyapunov–Schmidt method (see [1–3]). We consider the following evolution equation with quadratic non-linearity

$$\partial u / \partial t + R_0 \mathrm{A}_0 u + \varepsilon \mathrm{A} u + \mathrm{B}(u, u) = 0, \quad \varepsilon \in \mathbb{R} \tag{1.1}$$

where A_0 and A are linear and $\mathrm{B}(u, \upsilon)$ are bilinear bounded operators acting from a real Hilbert space G to a real Hilbert space H such that $G \subset H$, and where A and B depend on $\mu \in \mathbb{R}^n$. Looking for solutions of problem (1.1) that are $2\pi/c$-periodic in time, we obtain

$$c\partial u / \partial t + R_0 \mathrm{A}_0 u + \varepsilon \mathrm{A} u + \mathrm{B}(u, u) = 0 \tag{1.2}$$

We denote by $G_{2\pi}$ and $H_{2\pi}$ the spaces of square integrable functions with values in G and H, respectively, and by $G_{2\pi}^{\mathbb{C}}$ and $H_{2\pi}^{\mathbb{C}}$ the complexifications of these spaces. The scalar products in $H_{2\pi}^{\mathbb{C}}$ will be denoted by

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$$\langle u(t),\ v(t)\rangle_{H_{2\pi}^{\mathbb{C}}} = \frac{1}{2\pi}\int_0^{2\pi}(u(t),\ v(t))_{H^{\mathbb{C}}}\,dt$$

Proposition 1. The operator $Lu = c_0\partial u/\partial t + R_0A_0$ has a double semisimple eigenvalue $\lambda = 0$ with the corresponding eigenfunctions $\varphi_0 = e^{it}\chi_0$ and $\bar{\varphi}_0 = e^{-it}\bar{\chi}_0$, where $\chi_0 \in H^{\mathbb{C}}$.

Note that χ_0 is an eigenfunction of the eigenvalue problem $\lambda\chi + R_0A_0\chi = 0$ corresponding to the eigenvalue ic_0. Let ζ be an eigenfunction of the problem. Since $\lambda = 0$ is the semisimple eigenvalue, an eigenfunction $\psi_0 = e^{it}\zeta$ of the adjoint operator $L^*u = -c_0\partial u/\partial t + R_0A_0^*u$ exists such that

$$\langle \varphi_0,\ \psi_0\rangle = 1 \tag{1.3}$$

Let

$$Pu = u - Qu,\quad Q(u) = ½(\langle u,\ \psi_0\rangle\varphi_0 + \langle u,\ \bar{\psi}_0\rangle\bar{\varphi}_0),\quad u \in H_{2\pi}$$

(Q is the projection onto the kernel of L).

Proposition 2. L is a Fredholm operator, i.e. the equation $Lu = f$ is solvable for $f \in H_{2\pi}$ if and only if $\langle f, \psi_0\rangle = \langle u, \bar{\psi}_0\rangle = 0$, and L is an isomorphism of the spaces $PG_{2\pi}$ and $PH_{2\pi}$.

We write (1.2) in the form

$$Lu = (c_0 - c)du/dt - \varepsilon Au - B(u,\ u) \tag{1.4}$$

Then, applying P and Q to (1.4), we obtain the system of equations

$$LPu = -P\{\omega du/dt + \varepsilon Au + B(u,\ u)\} \tag{1.5}$$

$$LQu = -Q\{\omega du/dt + \varepsilon Au + B(u,\ u)\} \tag{1.6}$$

where $\omega = (c - c_0)$.

By Proposition 2, L realizes an isomorphism of the spaces $PG_{2\pi}$ and $PH_{2\pi}$. Consequently, putting $u = \upsilon + \gamma\mathrm{Re}\,e^{it}\chi_0$, where $\langle \upsilon, \bar{\psi}_0\rangle = 0$, from (1.5) we can find $\upsilon(\gamma,\omega,\varepsilon)$ using the implicit function theorem. Substituting the resulting expression into (1.6), we obtain the bifurcation equation

$$0 = \gamma f(c - c_0,\ \gamma^2,\ \varepsilon) = \gamma(i\omega + \varepsilon B + \gamma^2 D_0 + \gamma^4 D_1 + \omega\gamma^2 G + \ldots) \tag{1.7}$$

If

$$\mathrm{Re}D_0 \neq 0 \tag{1.8}$$

then by the implicit function theorem ω and ε can be given by series of the form

$$\omega = \sum_{n=1}^{\infty}\omega_n\gamma^{2n},\quad \varepsilon = \sum_{n=0}^{\infty}\varepsilon_n\gamma^{2n} \tag{1.9}$$

If we assume that

$$\mathrm{Re}D_0 \neq 0 \tag{1.10}$$

then

$$\gamma = \sum_{n=0}^{\infty}\gamma_n(\theta\varepsilon)^{(n+1)/2},\quad \omega = \sum_{n=1}^{\infty}c_{2n}(\theta\varepsilon)^n,\quad \theta = \pm 1 \tag{1.11}$$

where θ defines the direction of the bifurcation.

Note that the procedure for determining the constants $\{\gamma_{j-1};\ c_{2j}\}_{j=1}^{\infty}$ described earlier [6, 7] omits the stage when the implicit function theorem is applied to Eq. (1.5), and the above constants are determined along with the expansion

$$v = \sum_{j=0}^{\infty}v_j(\theta\varepsilon)^{(j+1)/2} \tag{1.12}$$

If $\mathrm{Re}D_0 = 0$ (and so $\varepsilon_1 = 0$) and $\varepsilon_2 \neq 0$, the expansions for γ and ω should be sought in terms of the powers of $(\theta\varepsilon)^{1/4}$, $\theta = \pm 1$. Condition (1.4) means that for an eigenvalue $\lambda(\varepsilon)$ of the problem

$$\lambda\chi + (R_0 A_0 + \varepsilon A)\chi = 0 \tag{1.13}$$

satisfying the condition $\lambda(0) = ic_0$ the relation

$$d\lambda(\varepsilon)/d\varepsilon|_{\varepsilon=0} = B$$

holds.

When an additional real-valued parameter α appears in the problem, as, for example, in the case of the Poiseuille flow in a plane channel, and the condition

$$\mathrm{Re}\lambda(\varepsilon, \alpha) = 0$$

defines a neutral curve on the plane (ε, α), condition (1.8) for a fixed α_* means that the straight line $\alpha = \alpha_*$ intersects the neutral curve transversally.

Obviously, condition (1.8) may also be violated. In particular, for Poiseuille flow in a plane channel this condition is violated at the maximum point of the neutral curve $(R_1, \alpha_1) \approx (8600, 1.097311)$. In this case the wave number α is a natural bifurcation parameter. The dependence of the equations of motion on α turns out to be quite complicated (cf. [6]) and the general formulae from which to obtain expansions of type (1.9) are somewhat less effective.

It is well known that for Poiseuille flow in a plane channel, condition (1.10) is violated at the point $(R_3, \alpha_3) \approx (5842, 197, 0.906672976)$ [7, 8]. Since $\varepsilon_2(\alpha_3 \neq 0)$, introducing a small parameter $h = \alpha - \alpha_3$, one can rewrite the second equality in (1.9) in the following form by the Weierstrass preparation theorem

$$\varepsilon - \sum_{j=1}^{\infty} \varepsilon_j(h)\gamma^{2j} = d(\gamma^2, \varepsilon, h)[H_0(\varepsilon, h) + H_1(\varepsilon, h)\gamma^2 + \varepsilon_2(\alpha_3)\gamma^4]$$

Here d, H_0 and H_1 are analytic functions uniquely defined by (1.9), such that $d(0, 0, 0) = 1$ and $H_0(0, 0) = H_1(0, 0) = 0$.

The equation

$$H_0(\varepsilon, h) + H_1(\varepsilon, h)\gamma^2 + \varepsilon_2(\alpha_3)\gamma^4 = 0 \tag{1.14}$$

which is equivalent to the second equation in (1.9) near zero, enables us to find a fold in the set of solutions of problem (1.2). Therefore the analysis of degeneracies in phase space provides additional information on the branching solution, which is completely lost if one considers bifurcations occurring under the variation of a single parameter with the remaining parameters fixed.

In the case of higher order degeneracies it is possible for branching solutions (secondary flows) to exhibit even more complex behaviour. For finite dimensional problems an analysis of various bifurcations up to degeneracies of codimension three has been carried out in [9, 10]. However, to apply these results it is necessary to know the coefficients in the second expansion (1.9).

2. AN ALGORITHM FOR COMPUTING THE COEFFICIENTS IN THE LYAPUNOV–SCHMIDT SERIES

According to the above discussion, the solution of problem (1.2) can be written as

$$u = v + \gamma\mathrm{Re}\varphi_0, \langle v, \psi\rangle = 0, \gamma \in \mathbb{R}^+$$

and u, c and ε can be expanded in a power series in γ. Putting $\upsilon_0 = \mathrm{Re}\,\varphi_0$ to make the formulae more symmetric, we shall determine the expansions

$$v = \sum_{n=0}^{\infty} v_n\gamma^{n+1}, \quad c - c_0 = \sum_{n=1}^{\infty} \omega_n\gamma^n, \quad \varepsilon = \sum_{n=1}^{\infty} \varepsilon_n\gamma^n \tag{2.1}$$

from the sequence of problems

$$Lv_n = \Phi_n, n = 0, 1, \ldots \tag{2.2}$$

$$\Phi_n = -\left\{ \sum_{j=0}^{n} \left(\omega_{n-j} \frac{\partial v_j}{\partial t} + \varepsilon_{n-j} A v_j \right) + \sum_{j=0}^{n-1} B\,(v_j,\ v_{n-j-1}) \right\}$$

By Proposition 2

$$\langle \Phi_n, \psi_0 \rangle = 0 \tag{2.3}$$

is a solvability condition for (2.2). For $n = 0$ we obtain the linearized problem

$$c_0 \partial v_0/\partial t + R_0 A_0 v_0 = 0$$

and for $n = 1$ we have

$$Lv_1 = -\{\omega_1 \partial v_0 / \partial t + \varepsilon A v_0 + B(v_0,\ v_0) + B(v_0, \bar{v}_0)\}$$

Using the condition $B = \mathrm{Re}\langle A\varphi_0, \psi_0 \rangle \neq 0$, we find $\varepsilon = \omega_1 = 0$. If we use the fact that L is a Fredholm operator, the solvability conditions for (2.2) yield, by induction, that $\varepsilon_{2n+1} = \omega_{2n+1} = 0$ and v_n is a trigonometric polynomial in t of degree $n + 1$, which is even if $n = 2l + 1$ and odd if $n = 2l$. Consequently

$$v_n = \mathrm{Re}\, w_n, \quad w_n = \sum_{k=0}^{n+1} e^{ikt} w_{nk}$$

$$\Phi_n = \mathrm{Re}\, F_n, \quad F_n = \sum_{k=0}^{n+1} e^{ikt} F_{nk}, \quad n = 0,\ 1, \ldots$$

Therefore the solvability condition (2.3) can be reduced to

$$(F_{n1},\ \zeta)_{H^C} = 0 \tag{2.4}$$

The key observation is that if $\mathrm{Im}\, F_{n0} = \mathrm{Im}\, w_{n0} = 0$ is fixed, the complex trigonometric polynomials F_n and w_n are uniquely defined. To verify this, we use the fact that

$$B(v_k,\ v_{n-k-1}) = \tfrac{1}{2} \mathrm{Re}(B(w_k,\ w_{n-k-1}) + B(w_k,\ \bar{w}_{n-k-1}))$$

in the following elementary assertion.

Lemma. Let Φ_{n-1} be a given real-valued trigonometric polynomial of degree n such that

$$\Phi_{n-1} = \mathrm{Re} \sum_{k=-n}^{n} a_k e^{ikt}, \quad a_0 = 0$$

Then there is a unique complex-valued trigonometric polynomial $F_{n-1} = \Sigma_{k=1}^{n} F_{n-1,k} e^{ikt}$ such that $\mathrm{Re}\, F_{n-1} = \Phi_{n-1}$, where $F_{n-1,k} = a_k + \bar{a}_{-k}$.

To prove the lemma it suffices to observe that

$$\mathrm{Re} \sum_{k=-n}^{n} a_k e^{ikt} = \mathrm{Re} \sum_{k=1}^{n} (a_k + \bar{a}_{-k}) e^{ikt}$$

Now from

$$\Phi_n = -\mathrm{Re}\left\{ \sum_{j=0}^{n} \left(\omega_{n-j} \frac{\partial w_i}{\partial t} + \varepsilon_{n-j} A w_j \right) + \frac{1}{2} \sum_{j=0}^{n-1} B(w_j,\ w_{n-j-1}) + B(w_j,\ \bar{w}_{n-j-1}) \right\}$$

we obtain

$$F_{nk} = -\left\{ \frac{1}{2} \sum_{j=0}^{n-1} (p_{nk}^j + q_{nk}^j) + \sum_{j=0}^{n-1} \varepsilon_{n-j} A w_{jk} + \sum_{j=0}^{n-1} ij\omega_{n-j} w_{jk} \right\}$$

where

$$p^j_{nk} = \sum_{l=\max(0,k+j-n)}^{\min(j+1,k)} B(w_{jl},\ w_{n-j-1,k-l}),\quad q^j_{n0} = \mathrm{Re} \sum_{l=0}^{\min(n+1,n-j)} B(w_{jl},\ \overline{w}_{n-j-1,l})$$

and for $k \neq 0$

$$q^j_{nk} = \sum_{s=k}^{\min(j+1,n-j+k)} B(w_{jk},\ \overline{w}_{n-j-1,s-k}) + \sum_{s=0}^{\min(j+1,n-k-j)} B(\overline{w}_{jk},\ w_{n-j-1,s-k})$$

Therefore the problems

$$L_k w_{nk} = F_{nk},\quad n = 0,\ 1,\ldots;\quad k = 0,\ 1,\ldots,\ n+1;\quad L_k = ikc_0 + R_0 A_0 \tag{2.5}$$

for the complex-valued functions w_{nk} are uniquely defined. These problems are solvable for $k \neq 1$ by Proposition 1, and the solvability condition for $k = 1$ can be satisfied by choosing the constants $\{\omega_j, \varepsilon_j\}$.

For $n = 1$ the solvability condition gives the relation

$$i\omega_2 + \varepsilon_2 B + \tfrac{1}{2}(B_0(\overline{w}_0,\ w_{12}) + 2B_0(w_0,\ w_{10}),\ \psi_0) = 0 \tag{2.6}$$

where $B_0(u, \upsilon) = B(u, \upsilon) + B(\upsilon, u)$.

It is obvious that if a numerical algorithm is constructed which enables us to determine ω_2 and ε_2 once the appropriate functionals are computed, then the same algorithm can be used without any major modifications to determine any finite number of coefficients $\{\omega_j, \varepsilon_j\}_{j=2}^n$ successively using the method described above. Therefore, using the results obtained in [9, 10], one can investigate the behaviour of branching solutions in the neighbourhood of a degenerate point of as high an order as desired.

The limit of applicability of the method in question in specific hydrodynamical problems is defined by the accuracy of the numerical solution of the eigenvalue and boundary value problems (2.5).

3. DEGENERATE BIFURCATIONS IN THE KOLMOGOROV AND COUETTE–POISEUILLE PROBLEMS

In 1959 Kolmogorov proposed to consider a model problem on the viscous fluid flow in a plane channel subject to a sinusoidal external force, with the no-slip condition replaced by the periodicity condition along the y coordinate normal to the channel axis x.

An analysis of the linear stability of Kolmogorov's flow revealed [11, 12] that the minimum critical Reynolds number corresponds to the wave number $\alpha = 0$, the neutral curve is defined by a monotone increasing function over the interval $\alpha \in (0, 1)$, and the flow is absolutely stable for $\alpha > 1$.

On the other hand, for Kolmogorov flow with other external forces (generalized Kolmogorov flows) the situation may turn out to be different, and the study of this class of flows is certainly interesting [13]. We shall assume that the mean velocity $Q = (0, \delta)$ and external force $F = (\gamma f(y), 0)'$ are fixed, $f(y)$ being a trigonometric polynomial. Then the steady-state solution has the form $U_*(y) = (V(y), \delta)'$, where V satisfies the equation

$$\nu V''(y) - \delta V'(y) + \gamma f(y) = 0$$

Since γ is arbitrary, we can assume that $\| V \|_{L_2} = 1$ and we introduce the Reynolds number $R = \gamma/\nu$. Computations show that if $\delta = 0$, then $\alpha = 0$ corresponds to the minimum Reynolds number for any $f(y)$.

The classical Kolmogorov flow corresponds to a velocity profile $U_k(y) = (\gamma/\nu \sin y, \delta)'$ and $\delta = 0$. Our computations revealed that $R \to \infty$ as $\delta \to 1$ for a fixed $\alpha \in (0, 1)$. For $\delta > 1$ the Kolmogorov flow is stable for all Reynolds numbers.

For generalized Kolmogorov flows for $\delta > 1$ the neutral curve may take the characteristic shape for problems with no-slip conditions at the channel walls. For example, for the velocity profile $U(y) = \cos y + \sin 2y$ the dashed line in Fig. 1 shows the neutral curve for $\delta = 0$, the dash–dot line for $\delta = 0.8$ and the solid line for $\delta = 1.1$.

Computations indicate† that for the Kolmogorov flow with velocity profile $U_k(y)$ the curves $\varepsilon_1(\alpha, \delta) = 0$ (the dashed line in Fig. 2) and $\varepsilon_2(\alpha, \delta) = 0$ (the dash–dot line) have points of intersection. At these points a degenerate bifurcation of codimension three occurs. The solid line in Fig. 2 is the boundary of the domain outside of which Kolmogorov flow is stable in the linear approximation.

For the Couette–Poiseuille problem computations show that the curves $\varepsilon_1(\alpha, \delta) = 0$ and $\varepsilon_2(\alpha, \delta) = 0$ have two points of intersection: $A_1 = (\alpha_1, \delta_1) = (0.691398427, 0.069459369)$ and $A_2 = (\alpha_2, \delta_2) \approx (0.516880749, 0.141800789)$.

†Detailed data are available on request. E-mail: varin@spp.keldysh.ru.

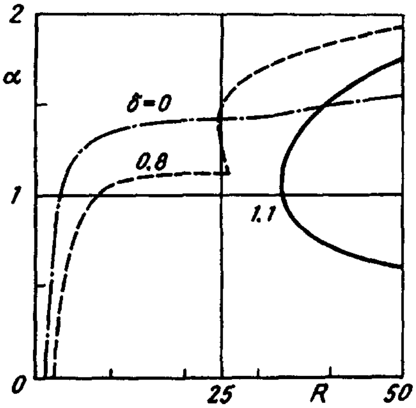

Fig. 1.

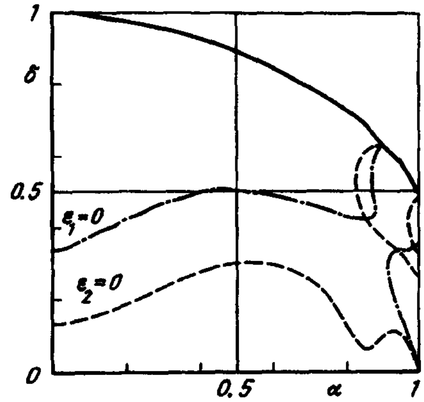

Fig. 2.

Let us recall that in the problem on the Couette–Poiseuille flow it is assumed, apart from a constant pressure gradient along the channel axis, that the velocity of the walls $u|_{y=\pm 1} = \pm V$ is also constant. On changing to dimensionless coordinates, this leads to the plane Couette–Poiseuille flow $u = (1 - y^2) + \delta y$ turning into plane Poiseuille flow at $\delta = 0$.

Let us return to expansion (1.9). The degeneracy $\varepsilon_1(\alpha_*, \delta_*) = 0$ means that for these parameter values the branching solution is given by a power series in $(\theta(R - R_0))^{1/4}$, $\theta = \pm 1$. If δ_* is fixed and the wave number α is considered as an additional bifurcation parameter, a fold occurs near the point $(R_0(\alpha_*, \delta_*), \alpha_*)$ in the case $\varepsilon_2(\alpha_*, \delta_*) \neq 0$, the projection of which onto the (α, R) plane is given by the discriminant curve of Eq. (1.14).

The inclusion of an additional bifurcation parameter δ into consideration leads, for example, to the fact that for each point of the neutral curve $(R_0(\alpha_*, \delta_*(\alpha))$ the branching solution can be expanded in a power series in $(\theta(R - R_0))^{1/4}$, $\theta = \pm 1$. If the value of (α_c, δ_c) corresponding to the intersection of the curves $\varepsilon_1(\alpha, \delta) = 0$ and $\varepsilon_2(\alpha, \delta)$ is fixed, the branching solution can be expanded in a power series in $(\theta(R - R_0))^{1/8}$, $\theta = \pm 1$, and the set of branching solutions for the parameters close to (α_c, δ_c) has a more complex structure than in the previous case and is described by a bicubical equation. More detailed information on this finite-dimensional problem can be found in [9, 10].

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